

SPECIAL POLYNOMIALS RELATED TO THE SUPERSYMMETRIC EIGHT-VERTEX MODEL. III. PAINLEVÉ VI EQUATION.

HJALMAR ROSENGREN

ABSTRACT. We prove that certain polynomials previously introduced by the author can be identified with tau functions of Painlevé VI, obtained from one of Picard's algebraic solutions by acting with a four-dimensional lattice of Bäcklund transformations. For particular lines in the lattice, this proves conjectures of Bazhanov and Mangazeev. As applications, we describe the behaviour of the corresponding solutions near the singular points of Painlevé VI, and obtain several new properties of our polynomials.

1. INTRODUCTION

The present work is the third part of a series, devoted to the study of special polynomials related to the eight-vertex model and other solvable lattice models of statistical mechanics. In [R3] we introduced, for each non-negative integer m , a four-dimensional lattice $T_n^{(k_0, k_1, k_2, k_3)}$ of symmetric rational functions in m variables, depending also on a parameter ζ . Here, k_j and n are integers, such that $m + \sum_j k_j = 2n$. The denominator in these functions is elementary, so they are essentially symmetric polynomials. For $m = 0$ and $m = 1$, polynomials corresponding to particular lines in the lattice appear in various ways in connection with solvable models [BM1, BM2, BH, FH, H, MB, RS, R1, R2, Z]. In [R4], we proved that the polynomials satisfy a non-stationary Schrödinger equation, which can be considered as the canonical quantization of Painlevé VI.

In the present work, we will show that the case $m = 0$ of the polynomials can be identified with tau functions of Painlevé VI, obtained from one of Picard's algebraic solutions by acting with a four-dimensional lattice of Bäcklund transformations. For particular lines in the lattice, this has been conjectured by Bazhanov and Mangazeev [BM2].

The plan of the paper is as follows. In §2, we recall the relevant facts on Painlevé VI. In particular, we must understand the action of Bäcklund transformations on tau functions. Although that topic has been considered by Masuda [M], his conventions are not ideal for our purposes and we therefore rederive some of his results in slightly different form. In §3, we consider the tau functions corresponding to one of Picard's algebraic solutions, realizing them explicitly as modular functions. After these preliminaries, we can turn to our main result, Theorem 4.2, which relates Painlevé tau functions to the case $m = 0$ of our polynomials. In §5 we give some applications. Using results of [R3] we describe the behaviour of the

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corresponding four-dimensional lattice of algebraic solutions to Painlevé VI at the singular points of the equation, see Corollary 5.2. We also obtain a new symmetry for our polynomials, Corollary 5.3. Reformulating bilinear identities for tau functions in terms of our polynomials, we can prove recursions along particular lines in the lattice conjectured by Bazhanov and Mangazeev [BM2, MB], see Proposition 5.4 and the subsequent discussion. Finally, we observe that the E_{VI} equation for the Hamiltonian of Painlevé VI leads to quadratic differential equations for our polynomials, see Proposition 5.5.

2. PAINLEVÉ VI

2.1. Bäcklund transformations. Painlevé VI is the differential equation

$$\begin{aligned} \frac{d^2 q}{dt^2} = & \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) \left(\frac{dq}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) \frac{dq}{dt} \\ & + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{q^2} + \gamma \frac{t-1}{(q-1)^2} + \delta \frac{t(t-1)}{(q-t)^2} \right). \end{aligned} \quad (2.1)$$

It is the most general Painlevé equation, and appears in many areas of contemporary mathematics and physics.

We will briefly review the rich symmetry theory of Painlevé VI. It is mainly due to Okamoto [O], although we will follow the exposition of Noumi and Yamada [NY]. We introduce parameters $\alpha_0, \dots, \alpha_4$ satisfying the constraint

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1 \quad (2.2)$$

and related to the parameters of (2.1) by

$$\alpha = \frac{\alpha_1^2}{2}, \quad \beta = -\frac{\alpha_4^2}{2}, \quad \gamma = \frac{\alpha_3^2}{2}, \quad \delta = \frac{1 - \alpha_0^2}{2}.$$

We let

$$\begin{aligned} H = & q(q-1)(q-t)p^2 - \{(\alpha_0 - 1)q(q-1) + \alpha_3 q(q-t) + \alpha_4(q-1)(q-t)\}p \\ & + \alpha_2(\alpha_1 + \alpha_2)(q-t). \end{aligned}$$

Then, (2.1) is equivalent to the Hamiltonian system

$$t(t-1) \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad t(t-1) \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \quad (2.3)$$

The system (2.3) admits many symmetries, or *Bäcklund transformations*. Indeed, it is invariant under the involutions s_j , r_j and t_j defined in the following table.

	α_0	α_1	α_2	α_3	α_4	q	p	t
s_0	$-\alpha_0$	α_1	$\alpha_2 + \alpha_0$	α_3	α_4	q	$p - \frac{\alpha_0}{q-t}$	t
s_1	α_0	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	α_4	q	p	t
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$\alpha_4 + \alpha_2$	$q + \frac{\alpha_2}{p}$	p	t
s_3	α_0	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	α_4	q	$p - \frac{\alpha_3}{q-1}$	t
s_4	α_0	α_1	$\alpha_2 + \alpha_4$	α_3	$-\alpha_4$	q	$p - \frac{\alpha_4}{q}$	t
r_1	α_1	α_0	α_2	α_4	α_3	$\frac{t(q-1)}{q-t}$	$\frac{(t-q)((q-t)p+\alpha_2)}{t(t-1)}$	t
r_3	α_3	α_4	α_2	α_0	α_1	$\frac{t}{q}$	$-\frac{q(pq+\alpha_2)}{t}$	t
t_1	α_0	α_4	α_2	α_3	α_1	$\frac{1}{q}$	$-q(pq + \alpha_2)$	$\frac{1}{t}$
t_3	α_0	α_1	α_2	α_4	α_3	$1 - q$	$-p$	$1 - t$

We will write $r_4 = r_1 r_3 = r_3 r_1$. We consider these symmetries as automorphisms of the differential field \mathcal{F}_0 generated by α_j , q , p and t , subject to the relation (2.2), and equipped with the derivation

$$\delta = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p} + t(t-1) \frac{\partial}{\partial t}.$$

In general, a Bäcklund transformation can be defined as a field automorphism σ such that

$$\sigma(\delta(x))\delta(\sigma(y)) = \sigma(\delta(y))\delta(\sigma(x)), \quad x, y \in \mathcal{F}_0.$$

Choosing, without loss of generality, $y = t$, we find that the Bäcklund property means that, for all k ,

$$s_k \circ \delta = \delta \circ s_k, \quad r_k \circ \delta = \delta \circ r_k, \quad (2.4a)$$

$$t_1 \circ \delta = \frac{1}{t} \delta \circ t_1, \quad t_3 \circ \delta = -\delta \circ t_3. \quad (2.4b)$$

The Bäcklund transformations defined above satisfy the relations

$$s_i^2 = 1, \quad i = 0, 1, 2, 3, 4, \quad (2.5a)$$

$$(s_i s_j)^2 = (s_i s_2)^3 = 1, \quad i, j = 0, 1, 3, 4, \quad (2.5b)$$

$$r_1^2 = r_3^2 = (r_1 r_3)^2 = 1, \quad (2.5c)$$

$$r_1 s_{0,1,2,3,4} = s_{1,0,2,4,3} r_1, \quad r_3 s_{0,1,2,3,4} = s_{3,4,2,0,1} r_3, \quad (2.5d)$$

$$t_1^2 = t_3^2 = (t_1 t_3)^3 = 1, \quad (2.5e)$$

$$t_1 s_{0,1,2,3,4} = s_{0,4,2,3,1} t_1, \quad t_3 s_{0,1,2,3,4} = s_{0,1,2,4,3} t_3, \quad (2.5f)$$

$$t_1 r_{1,3,4} = r_{4,3,1} t_1, \quad t_3 r_{1,3,4} = r_{1,4,3} t_3. \quad (2.5g)$$

In particular, $(s_j)_{j=0}^4$ generate the D_4 affine Weyl group. Adjoining $(r_j)_{j=1,3}$ gives the *extended* D_4 affine Weyl group. The elements t_j generate the symmetric group S_3 ; adjoining them gives the extended F_4 affine Weyl group.

The group of Bäcklund transformations contains a subgroup isomorphic to \mathbb{Z}^4 , corresponding to the D_4 weight lattice. It is generated by the mutually commuting

elements

$$\begin{aligned} T_1 &= r_1 s_1 s_2 s_3 s_4 s_2 s_1, & T_2 &= s_0 s_2 s_1 s_3 s_4 s_2 s_1 s_3 s_4 s_2, \\ T_3 &= r_3 s_3 s_2 s_1 s_4 s_2 s_3, & T_4 &= r_4 s_4 s_2 s_1 s_3 s_2 s_4. \end{aligned}$$

We will need the commutation relations

$$s_0 T_j = \begin{cases} T_j T_2^{-1} s_0, & j = 1, 3, 4, \\ T_2^{-1} s_0, & j = 2, \end{cases} \quad (2.6a)$$

$$s_i T_j = \begin{cases} T_2 T_i^{-1} s_i, & i = j = 1, 3, 4, \\ T_j s_i, & i = 1, 3, 4, j = 1, 2, 3, 4, i \neq j. \end{cases} \quad (2.6b)$$

$$s_2 T_j = \begin{cases} T_j s_2, & j = 1, 3, 4, \\ T_1 T_2^{-1} T_3 T_4 s_2, & j = 2. \end{cases} \quad (2.6c)$$

2.2. Tau functions. If one computes the action of some element of \mathbb{Z}^4 on the generator q , corresponding to a solution, one finds that it always factors. For instance,

$$\begin{aligned} T_3(q) &= \frac{t(p(t-q) + \alpha_0)}{qp(t-q) - \alpha_2 q + (\alpha_0 + \alpha_2)t} \\ &\quad \times \frac{qp(t-q) + (\alpha_0 + \alpha_4)q - \alpha_4 t}{qp(t-q) - (\alpha_1 + \alpha_2)q + (\alpha_0 + \alpha_1 + \alpha_2)t}. \end{aligned} \quad (2.7)$$

The non-trivial factors are essentially *tau functions*. To incorporate these, we need to work in a field extension of \mathcal{F}_0 . One way to do this was proposed by Masuda [M]. It is, however, not ideal for our purposes and we will therefore work with a variation of Masuda's construction.

We introduce the modified Hamiltonian

$$\begin{aligned} h_0 &= H + \frac{t}{12} (2(\alpha_0 - 1)^2 - \alpha_1^2 + 2\alpha_3^2 - \alpha_4^2 + 6(\alpha_0 - 1)\alpha_3) \\ &\quad + \frac{t-1}{12} (2(\alpha_0 - 1)^2 - \alpha_1^2 - \alpha_3^2 + 2\alpha_4^2 + 6(\alpha_0 - 1)\alpha_4). \end{aligned} \quad (2.8)$$

Note that (2.3) holds with H replaced by h_0 . The extra terms have been introduced so that

$$s_1(h_0) = s_2(h_0) = s_3(h_0) = s_4(h_0) = h_0, \quad (2.9)$$

$$t_1(h_0) = \frac{h_0}{t}, \quad t_3(h_0) = -h_0. \quad (2.10)$$

(Masuda [M] works with a different modification that satisfies (2.9) but not (2.10).) We also define

$$h_1 = r_1(h_0), \quad h_3 = r_3(h_0), \quad h_4 = r_4(h_0), \quad h_2 = h_1 + s_1(h_1) - \frac{t}{3} + \frac{1}{6}.$$

We denote by \mathcal{F} the field extension of \mathcal{F}_0 by the additional generators $u, v, \tau_0, \dots, \tau_4$. The generators u and v satisfy

$$t = u^2 v^4, \quad 1 - t = u^4 v^2 \quad (2.11)$$

and thus formally correspond to the roots $t^{-1/6}(1-t)^{1/3}$ and $t^{1/3}(1-t)^{-1/6}$. We extend δ to the new generators by

$$\delta(u) = \frac{u(t+1)}{6}, \quad \delta(v) = \frac{v(t-2)}{6}, \quad \delta(\tau_j) = \tau_j h_j, \quad j = 0, \dots, 4,$$

which is consistent with (2.11). Finally, we extend the action of the Bäcklund transformations by the following table.

	u	v	τ_0	τ_1	τ_2	τ_3	τ_4
s_0	u	v	$\frac{i(t-q)\tau_2}{u^2 v^2 \tau_0}$	τ_1	τ_2	τ_3	τ_4
s_1	u	v	τ_0	$\frac{iuv\tau_2}{\tau_1}$	τ_2	τ_3	τ_4
s_2	u	v	τ_0	τ_1	$\frac{p\tau_0\tau_1\tau_3\tau_4}{\tau_2}$	τ_3	τ_4
s_3	u	v	τ_0	τ_1	τ_2	$\frac{(1-q)\tau_2}{u\tau_3}$	τ_4
s_4	u	v	τ_0	τ_1	τ_2	τ_3	$\frac{q\tau_2}{v\tau_4}$
r_1	u	$-v$	τ_1	τ_0	$\frac{(q-t)\tau_2}{u^3 v^3}$	τ_4	τ_3
r_3	$-u$	v	τ_3	τ_4	$\frac{iq\tau_2}{uv^2}$	τ_0	τ_1
t_1	u	$\frac{i}{uv}$	$i\tau_0$	τ_4	$-q\tau_2$	τ_3	τ_1
t_3	v	u	$i\tau_0$	τ_1	τ_2	τ_4	τ_3

It is straight-forward to check that this is consistent with (2.11) and that (2.4) hold on the new generators.

The relations (2.5a)–(2.5d) are all valid on \mathcal{F} . In particular, the operators T_j still define an action of \mathbb{Z}^4 . This is in contrast to the alternative definition of Masuda. However, the relations (2.5e)–(2.5g) are *not* valid.

The relations (2.5e) are replaced by

$$t_1^2 = t_3^2 = (t_1 t_3)^3 = \sigma, \quad \sigma^2 = 1, \quad (2.12)$$

where σ is the Bäcklund transformation mapping τ_0 to $-\tau_0$ and fixing all other generators. Equivalently, with $x = t_1$ and $a = t_1 t_3$,

$$a^6 = 1, \quad x^2 = a^3, \quad axa = x,$$

which are the standard defining relations for the dicyclic group Dic_3 of order 12.

One has a lot of freedom when extending the Bäcklund transformations to \mathcal{F} . Our definition has been adapted to the specific class of solutions that we will consider. For these solutions, (2.12) can be realized with t_j represented by changes of variable, see (3.8a). One can modify the definition of t_j so that the relations (2.5e) remain valid, but that would be inconvenient for our purposes.

One can write down modified versions of (2.5f)–(2.5g) that are valid on \mathcal{F} , but we do not do so here. However, we will need the commutation relations between t_j and the embedded lattice \mathbb{Z}^4 .

Lemma 2.1. *Let ψ_j and χ_j be the mutually commuting field automorphisms of \mathcal{F} fixing \mathcal{F}_0 and acting on the remaining generators according to the following table.*

	u	v	τ_0	τ_1	τ_2	τ_3	τ_4
ψ_1	$-u$	$-v$	$-\mathrm{i}\tau_0$	$-\mathrm{i}\tau_1$	τ_2	$-\mathrm{i}\tau_3$	$-\mathrm{i}\tau_4$
ψ_2	u	v	τ_0	$-\tau_1$	τ_2	$-\tau_3$	$-\tau_4$
ψ_3	u	$-v$	$-\mathrm{i}\tau_0$	$-\mathrm{i}\tau_1$	τ_2	$-\mathrm{i}\tau_3$	$-\mathrm{i}\tau_4$
ψ_4	$-u$	v	$-\mathrm{i}\tau_0$	$-\mathrm{i}\tau_1$	τ_2	$-\mathrm{i}\tau_3$	$-\mathrm{i}\tau_4$
χ_1	u	v	τ_0	τ_1	τ_2	$-\tau_3$	$-\tau_4$
χ_2	u	v	$-\tau_0$	$-\tau_1$	τ_2	$-\tau_3$	$-\tau_4$
χ_3	u	v	τ_0	$-\tau_1$	τ_2	τ_3	$-\tau_4$
χ_4	u	v	τ_0	$-\tau_1$	τ_2	$-\tau_3$	τ_4

Then, the following relations hold on \mathcal{F} :

$$t_1 T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} = T_1^{l_4} T_2^{l_2} T_3^{l_3} T_4^{l_1} t_1 X, \quad (2.13)$$

$$t_3 T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} = T_1^{l_1} T_2^{l_2} T_3^{l_4} T_4^{l_3} t_3 X, \quad (2.14)$$

where

$$X = \psi_1^{l_1} \psi_2^{l_2} \psi_3^{l_3} \psi_4^{l_4} \chi_1^{\binom{l_1}{2} + l_3 l_4} \chi_3^{\binom{l_3}{2} + l_1 l_4} \chi_4^{\binom{l_4}{2} + l_1 l_3} \times \chi_2^{l_2(l_1 + l_3 + l_4) + \frac{1}{2}(l_3 - l_1)(l_4 - l_1)(l_4 - l_3)}. \quad (2.15)$$

Proof. We prove (2.13) by induction on $\sum_j |l_j|$, using the relations

$$t_1 T_{1,2,3,4} = T_{4,2,3,1} t_1 \psi_{1,2,3,4}, \quad (2.16)$$

$$\psi_j T_k = T_k \psi_j \cdot \begin{cases} 1, & j = k = 2, \\ \chi_l, & j = k = l \neq 2 \text{ or } \{j, k, l\} = \{1, 3, 4\}, \\ \chi_2, & \text{if exactly one of } j \text{ and } k \text{ equals } 2, \end{cases} \quad (2.17)$$

$$\chi_j T_k = T_k \chi_j \cdot \begin{cases} 1, & j = k \text{ or } 2 \in \{j, k\}, \\ \chi_2, & j \neq k \text{ and } 2 \notin \{j, k\}. \end{cases} \quad (2.18)$$

Assuming (2.13), we need to show that the same relation holds when l_j is replaced by $l_j \pm 1$ for some j . For instance, when l_1 is replaced by $l_1 \pm 1$, the induction step would follow from

$$T_4 t_1 X_{l_1+1} = t_1 X_{l_1} T_1,$$

where we indicate the dependence of X on l_1 . Using (2.16), this is reduced to

$$\psi_1^{-1} X_{l_1+1} = T_1^{-1} X_{l_1} T_1. \quad (2.19)$$

On the right-hand side of (2.19), we apply conjugation by T_1 to each factor in (2.15). Using (2.17) and (2.18), we obtain

$$(\psi_1\chi_1)^{l_1}(\psi_2\chi_2)^{l_2}(\psi_3\chi_4)^{l_3}(\psi_4\chi_3)^{l_4}\chi_1^{\binom{l_1}{2}+l_3l_4}(\chi_2\chi_3)^{\binom{l_3}{2}+l_1l_4}(\chi_2\chi_4)^{\binom{l_4}{2}+l_1l_3} \\ \times \chi_2^{l_2(l_1+l_3+l_4)+\frac{1}{2}(l_3-l_1)(l_4-l_1)(l_4-l_3)}.$$

Writing

$$\frac{(l_3-l_1)(l_4-l_1)(l_4-l_3)}{2} = \binom{l_1}{2}(l_4-l_3) + \binom{l_3}{2}(l_1-l_4) + \binom{l_4}{2}(l_3-l_1),$$

this is seen to equal the left-hand side of (2.19). More generally, the induction step in l_j can be reduced to $\psi_j^{-1}X_{l_j+1} = T_j^{-1}X_{l_j}T_j$, which for $j = 3$ and 4 follows from (2.19) by symmetry, and for $j = 2$ is proved similarly. Moreover, since

$$t_3T_{1,2,3,4} = T_{1,2,4,3}t_3\psi_{1,2,3,4},$$

the same arguments prove (2.14). \square

We are interested in the lattice of tau functions

$$\tau_{l_1l_2l_3l_4} = T_1^{l_1}T_2^{l_2}T_3^{l_3}T_4^{l_4}\tau_0. \quad (2.20)$$

Corollary 2.2. *The transformations t_j act on (2.20) by*

$$t_1(\tau_{l_1l_2l_3l_4}) = i^{1+(2l_2-1)(l_1+l_3+l_4)+(l_3-l_1)(l_4-l_1)(l_4-l_3)}\tau_{l_4l_2l_3l_1}, \\ t_3(\tau_{l_1l_2l_3l_4}) = i^{1+(2l_2-1)(l_1+l_3+l_4)+(l_3-l_1)(l_4-l_1)(l_4-l_3)}\tau_{l_1l_2l_4l_3}$$

It will be convenient to introduce an index l_0 , determined from the other l_j by

$$l_0 + l_1 + 2l_2 + l_3 + l_4 = 0. \quad (2.21)$$

Lemma 2.3. *With $\mathbf{T} = T_1^{l_1}T_2^{l_2}T_3^{l_3}T_4^{l_4}$, we have*

$$\begin{aligned} \mathbf{T}(\alpha_j) &= \alpha_j - l_j, \quad j = 0, \dots, 4, \\ \mathbf{T}(u) &= (-1)^{l_3+l_4}u, \\ \mathbf{T}(v) &= (-1)^{l_1+l_4}v, \\ \mathbf{T}(q) &= (-1)^{l_3+l_4} \frac{iuv^2\tau_{l_1,l_2,l_3,l_4+1}\tau_{l_1,l_2+1,l_3,l_4-1}}{\tau_{l_1+1,l_2,l_3,l_4}\tau_{l_1-1,l_2+1,l_3,l_4}}, \\ \mathbf{T}(p) &= -\frac{\tau_{l_1+1,l_2,l_3,l_4}\tau_{l_1-1,l_2+1,l_3,l_4}\tau_{l_1,l_2-1,l_3+1,l_4+1}}{u^2v^2\tau_{l_1,l_2,l_3,l_4}\tau_{l_1,l_2,l_3+1,l_4}\tau_{l_1,l_2,l_3,l_4+1}}. \end{aligned} \quad (2.22)$$

This can be checked by direct computation. To prove the identities for $\mathbf{T}(q)$ and $\mathbf{T}(p)$ one first verifies them for $\mathbf{T} = \text{id}$ and then act on both sides with \mathbf{T} .

As an example, when $\mathbf{T} = T_3$, Lemma 2.3 gives

$$T_3(q) = -\frac{iuv^2\tau_{0,0,1,1}\tau_{0,1,1,-1}}{\tau_{1,0,1,0}\tau_{-1,1,1,0}},$$

where we can compute

$$\begin{aligned}
\tau_{0,0,1,1} &= -\frac{\tau_3\tau_4}{uv\tau_0} (p(t-q) + \alpha_0), \\
\tau_{0,1,1,-1} &= -\frac{\tau_2\tau_3}{uv^2\tau_0\tau_4} (pq(t-q) + (\alpha_0 + \alpha_4)q - \alpha_4t), \\
\tau_{1,0,1,0} &= \frac{i\tau_1\tau_3v}{t\tau_0} (pq(t-q) - \alpha_2q + (\alpha_0 + \alpha_2)t), \\
\tau_{-1,1,1,0} &= -\frac{\tau_2\tau_3}{uv^2\tau_0\tau_1} (pq(t-q) - (\alpha_0 + \alpha_2)q + (\alpha_0 + \alpha_1 + \alpha_2)t).
\end{aligned}$$

Thus, we recover the factorization (2.7).

2.3. Bilinear identities. Tau functions of Painlevé VI satisfy many bilinear identities [M]. We do not give an exhaustive list, but state a few examples that we need.

Proposition 2.4. *The tau functions (2.20) satisfy the bilinear identities*

$$\begin{aligned}
&(l_0 + l_2 + l_3 - \alpha_0 - \alpha_2 - \alpha_3) \delta(\tau_{l_1, l_2, l_3, l_4}) \tau_{l_1, l_2+1, l_3-1, l_4} \\
&\quad + (l_1 + l_2 + l_4 - \alpha_1 - \alpha_2 - \alpha_4) \tau_{l_1, l_2, l_3, l_4} \delta(\tau_{l_1, l_2+1, l_3-1, l_4}) \\
&\quad + Q(l_0 - \alpha_0, l_1 - \alpha_1, l_3 - \alpha_3, l_4 - \alpha_4) \tau_{l_1, l_2, l_3, l_4} \tau_{l_1, l_2+1, l_3-1, l_4} \\
&= u^2 v^2 \tau_{l_1+1, l_2, l_3-1, l_4+1} \tau_{l_1-1, l_2+1, l_3, l_4-1},
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&(l_0 + l_2 + l_3 - \alpha_0 - \alpha_2 - \alpha_3) \delta(\tau_{l_1, l_2, l_3, l_4}) \tau_{l_1+1, l_2-1, l_3, l_4+1} \\
&\quad + (l_1 + l_2 + l_4 - \alpha_1 - \alpha_2 - \alpha_4) \tau_{l_1, l_2, l_3, l_4} \delta(\tau_{l_1+1, l_2-1, l_3, l_4+1}) \\
&\quad + R(l_0 - \alpha_0, l_1 - \alpha_1, l_3 - \alpha_3, l_4 - \alpha_4) \tau_{l_1, l_2, l_3, l_4} \tau_{l_1+1, l_2-1, l_3, l_4+1} \\
&= u^2 v^2 \tau_{l_1, l_2-1, l_3+1, l_4} \tau_{l_1+1, l_2, l_3-1, l_4+1},
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
&\frac{\delta^2(\tau_{l_1 l_2 l_3 l_4}) \tau_{l_1 l_2 l_3 l_4}}{t} - \frac{\delta(\tau_{l_1 l_2 l_3 l_4})^2}{t} - \delta(\tau_{l_1 l_2 l_3 l_4}) \tau_{l_1 l_2 l_3 l_4} \\
&\quad + S(l_0 - \alpha_0, l_1 - \alpha_1, l_3 - \alpha_3, l_4 - \alpha_4) \tau_{l_1 l_2 l_3 l_4}^2 \\
&= (-1)^{l_3+l_4} iu \tau_{l_1, l_2+1, l_3-1, l_4} \tau_{l_1, l_2-1, l_3+1, l_4},
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
Q(l_0, l_1, l_3, l_4) &= \frac{1}{12} (l_1 - l_4)(l_0 - l_3 + 1)(t - 2) \\
&\quad + \frac{1}{12} \left((l_1 + l_4)^2 + \left(l_0 - l_1 + \frac{1}{2} \right) \left(l_3 - l_4 - \frac{1}{2} \right) - \frac{1}{4} \right) (t + 1), \\
R(l_0, l_1, l_3, l_4) &= \frac{1}{12} (l_1 - l_4)(l_0 - l_3 + 1)(t - 2) \\
&\quad + \frac{1}{12} \left((l_0 + l_3 + 1)^2 + \left(l_0 - l_1 + \frac{1}{2} \right) \left(l_3 - l_4 - \frac{1}{2} \right) - \frac{1}{4} \right) (t + 1), \\
S(l_0, l_1, l_3, l_4) &= \frac{1}{12} (2l_0^2 - l_1^2 + 2l_3^2 - l_4^2 + 6l_0l_3 + 4l_0 + 6l_3 + 2).
\end{aligned}$$

Proof. It is enough to prove these identities when $l_j = 0$ for all j , since the general case follows using Lemma 2.3. In that case, (2.23) takes the form

$$\begin{aligned} & -(\alpha_0 + \alpha_2 + \alpha_3)(h_0) - (\alpha_1 + \alpha_2 + \alpha_4)(T_2 T_3^{-1})(h_0) + Q(-\alpha_0, -\alpha_1, -\alpha_3, -\alpha_4) \\ & = u^2 v^2 \frac{\tau_{1,0,-1,1} \tau_{-1,1,0,-1}}{\tau_{0,1,-1,0} \tau_{0,0,0,0}}. \end{aligned} \quad (2.26)$$

This can be checked by a straight-forward computation, which simplifies if one notes that, by (2.5d) and (2.9), $(T_2 T_3^{-1})(h_0) = (r_3 s_0)(h_0)$. Acting with s_2 on (2.26), using (2.6c) and (2.9), gives the case $l_j \equiv 0$ of (2.24). The case $l_j \equiv 0$ of (2.25) can be written

$$\frac{\delta(h_0)}{t} - h_0 + S(-\alpha_0, -\alpha_1, -\alpha_3, -\alpha_4) = iu \frac{\tau_{0,1,-1,0} \tau_{0,-1,1,0}}{\tau_{0,0,0,0}^2},$$

which can again be verified directly. \square

2.4. Differential equations for tau functions. Painlevé VI can be reformulated as a differential equation for the Hamiltonian, known as the E_{VI} equation [JM, O]. In terms of the parameters

$$b_1 = \frac{\alpha_3 + \alpha_4}{2}, \quad b_2 = \frac{\alpha_4 - \alpha_3}{2}, \quad b_3 = \frac{\alpha_0 + \alpha_1 - 1}{2}, \quad b_4 = \frac{\alpha_0 - \alpha_1 - 1}{2}$$

it takes the form

$$\begin{aligned} & \frac{dh}{dt} \left(t(t-1) \frac{d^2 h}{dt^2} \right)^2 + \left(\frac{dh}{dt} \left(2h - (2t-1) \frac{dh}{dt} \right) + b_1 b_2 b_3 b_4 \right)^2 \\ & = \prod_{k=1}^4 \left(\frac{dh}{dt} + b_k^2 \right), \end{aligned} \quad (2.27)$$

where h is related to (2.8) by

$$h = h_0 - \frac{C}{24} (2t-1), \quad (2.28)$$

with

$$C = (\alpha_0 - 1)^2 + \alpha_1^2 + \alpha_3^2 + \alpha_4^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2). \quad (2.29)$$

Expressing h in terms of tau functions, (2.27) takes a rather complicated form. To obtain a simpler identity, we first cancel the term $\prod_k b_k^2$ and the factor dh/dt on both sides, then differentiate in t and finally cancel the factor $d^2 h/dt^2$. This leads to the alternative differential equation

$$\begin{aligned} & t^2(t-1)^2 \frac{d^3 h}{dt^3} + t(t-1)(2t-1) \frac{d^2 h}{dt^2} + 6t(t-1) \left(\frac{dh}{dt} \right)^2 + 4(1-2t)h \frac{dh}{dt} \\ & - \left(\sum_{k=1}^4 b_k^2 \right) \frac{dh}{dt} + 2h^2 + (1-2t)b_1 b_2 b_3 b_4 - \frac{1}{2} \sum_{1 \leq j < k \leq 4} b_j^2 b_k^2 = 0. \end{aligned}$$

Substituting (2.28) and writing $\delta = t(t-1)d/dt$ gives

$$\begin{aligned} & \delta^3(h_0) + 2(1-2t)\delta^2(h_0) + 6\delta(h_0)^2 + 4(1-2t)h_0\delta(h_0) \\ & - \frac{(C-6)t(t-1) + C-3}{3} \delta(h_0) + 2t(t-1)h_0^2 + \frac{Ct(t-1)(2t-1)}{6} h_0 \\ & - \frac{t(t-1)G}{8} = 0, \end{aligned} \quad (2.30)$$

where

$$\begin{aligned} G = & (\alpha_4 - \alpha_3)(\alpha_3 + \alpha_4)(\alpha_0 + \alpha_1 - 1)(\alpha_0 - \alpha_1 - 1)t \\ & + (\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)(\alpha_0 + \alpha_4 - 1)(\alpha_0 - \alpha_4 - 1). \end{aligned} \quad (2.31)$$

One may check directly that (2.30) holds as an identity in \mathcal{F}_0 .

Substituting $h_0 = \delta(\tau_0)/\tau_0$ in (2.30), all terms with denominators τ_0^3 or τ_0^4 cancel. After simplification, we find that $\tau = \tau_0$ satisfies the equation

$$\begin{aligned} & \delta^4(\tau)\tau - 4\delta^3(\tau)\delta(\tau) + 2(1-2t)\delta^3(\tau)\tau + 3\delta^2(\tau)^2 - 2(1-2t)\delta^2(\tau)\delta(\tau) \\ & - \frac{(C-6)t(t-1) + C-3}{3} \delta^2(\tau)\tau + \frac{C(t^2-t+1)-3}{3} \delta(\tau)^2 \\ & + \frac{Ct(t-1)(2t-1)}{6} \delta(\tau)\tau - \frac{t(t-1)G}{8} \tau^2 = 0. \end{aligned} \quad (2.32)$$

Acting by \mathbb{Z}^4 , we obtain the following result, which we have not found in the literature. Analogous results for other Painlevé equations are discussed in [C].

Proposition 2.5. *The tau functions $\tau = \tau_{l_1 l_2 l_3 l_4}$ satisfy (2.32), with C and G obtained from (2.29) and (2.31) by replacing each α_j with $\alpha_j - l_j$.*

3. SEED SOLUTION

3.1. An algebraic Picard solution. When

$$\alpha_0 = \alpha_1 = \alpha_3 = \alpha_4 = 0, \quad \alpha_2 = \frac{1}{2},$$

Painlevé VI can be solved explicitly in terms of Weierstrass's \wp -function. This was done by Picard already in 1889 [P]. The general solution is labelled by two complex parameters ν_1, ν_2 ; it is algebraic if $\nu_1, \nu_2 \in \mathbb{Q}$ [Ma]. In [BM2], the solution with $(\nu_1, \nu_2) = (1, 1/3)$ was expressed as

$$q^4 - 4tq^3 + 6tq^2 - 4tq + t^2 = 0. \quad (3.1)$$

We are interested in the corresponding lattice of tau functions (2.20).

We will substitute

$$t(\tau) = \frac{\theta(-1; p^6)^4}{\theta(-p^3; p^6)^4}, \quad p = e^{\pi i \tau},$$

where τ is in the upper half-plane and

$$\theta(x; p) = \prod_{j=0}^{\infty} (1 - p^j x) \left(1 - \frac{p^{j+1}}{x} \right).$$

We claim that t is a *modular function* for $\Gamma_0(6, 2)$, that is, it is meromorphic and invariant under the modular transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \quad (3.2)$$

such that $b \equiv 0 \pmod{2}$ and $c \equiv 0 \pmod{6}$. To see this is, we use [R2, Lemma 9.1] to write

$$t = \frac{\zeta(\zeta + 2)^3}{(1 + 2\zeta)^3}, \quad (3.3)$$

where

$$\zeta = \frac{\omega^2 \theta(-1; p^2) \theta(-p\omega; p^2)}{\theta(-p; p^2) \theta(-\omega; p^2)}, \quad \omega = e^{2\pi i/3}.$$

As is explained in [R3, §2.9], ζ is a *Hauptmodul* for $\Gamma_0(6, 2)$, which means that it generates the corresponding field of modular functions. Note that the values

$$\zeta = 0, -1, 1, -2, -\frac{1}{2}, \infty \quad (3.4)$$

at the six cusps of $\Gamma_0(6, 2)$ correspond precisely to the three singular points $t = 0, 1, \infty$ of (2.1).

Making the change of variables (3.3) in (3.1), we find that there is a rational solution in ζ (and thus modular in τ) given by

$$q = \frac{\zeta(\zeta + 2)}{2\zeta + 1}.$$

It is then easy to solve (2.3), giving

$$p = \frac{2\zeta + 1}{2(1 - \zeta)(\zeta + 2)}.$$

We will write δ for the differentiation

$$\delta = t(t - 1) \frac{d}{dt} = \frac{\zeta(\zeta + 1)(\zeta - 1)(\zeta + 2)}{2(2\zeta + 1)^2} \frac{d}{d\zeta} \quad (3.5)$$

acting on $\mathbb{C}(\zeta)$. The fact that (3.1) solves (2.1) can then be formulated as

$$\mathbf{X} \circ \delta = \delta \circ \mathbf{X}, \quad (3.6)$$

where $\mathbf{X} : \mathcal{F}_0 \rightarrow \mathbb{C}(\zeta)$ is the field automorphism defined by

$$\begin{aligned} \mathbf{X}(\alpha_0) &= \mathbf{X}(\alpha_1) = \mathbf{X}(\alpha_3) = \mathbf{X}(\alpha_4) = 0, & \mathbf{X}(\alpha_2) &= \frac{1}{2}, \\ \mathbf{X}(q) &= \frac{\zeta(\zeta + 2)}{2\zeta + 1}, & \mathbf{X}(p) &= \frac{2\zeta + 1}{2(1 - \zeta)(\zeta + 2)}, & \mathbf{X}(t) &= \frac{\zeta(\zeta + 2)^3}{(1 + 2\zeta)^3}. \end{aligned}$$

We remark that, in terms of the alternative Hauptmodul $\bar{\gamma} = (1 - \zeta)/(1 + \zeta)$,

$$t = \frac{(1 - \bar{\gamma})(3 + \bar{\gamma})^3}{(1 + \bar{\gamma})(3 - \bar{\gamma})^3}, \quad q = \frac{(1 - \bar{\gamma})(3 + \bar{\gamma})}{(1 + \bar{\gamma})(3 - \bar{\gamma})},$$

which is the parametrization of (3.1) used in [BM2]. Moreover, with $s = -\bar{\gamma}$ we have

$$\mathbf{X}(T_2^{-1}T_3^2r_4(q)) = \frac{3(3-s)(s+1)(s^2-3)^2}{(3+s)^2(s^6+3s^4-9s^2+9)},$$

which is the solution [D, (E.37)].

3.2. Modular tau functions. Although our seed solution is modular for $\Gamma_0(6, 2)$, the corresponding tau functions are only modular for a subgroup. We will now describe the field \mathcal{M} generated by all the modular functions that we need.

For $p = e^{\pi i \tau}$, Dedekind's eta function is given by

$$\eta(\tau) = p^{\frac{1}{12}} \prod_{j=0}^{\infty} (1 - p^{2(j+1)}).$$

It satisfies

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau), \quad \eta\left(\tau + \frac{1}{2}\right) = \frac{e^{\frac{\pi i}{24}} \eta(2\tau)^3}{\eta(\tau) \eta(4\tau)}. \quad (3.7)$$

In the notation

$$[a_1^{k_1}, \dots, a_m^{k_m}](\tau) = \eta(a_1\tau)^{k_1} \cdots \eta(a_m\tau)^{k_m},$$

we define \mathcal{M} to be the field generated by the five functions

$$\phi_1 = \frac{[(1/2)^2]}{[1^2]}, \quad \phi_2 = \frac{[2^2]}{[1^2]}, \quad \phi_3 = \frac{[3/2]}{[1/2]}, \quad \phi_4 = \frac{[6]}{[2]}, \quad \phi_5 = \frac{[3]}{[1]}.$$

It can be deduced from [GS], or checked directly using Corollary 3.3, that the functions ϕ_j are all invariant under the group $K \subseteq \Gamma_0(6, 2)$, consisting of transformations (3.2) with

$$a \equiv d \equiv \pm 1 \pmod{12}, \quad b \equiv 0 \pmod{24}, \quad c \equiv 0 \pmod{72}.$$

Thus, \mathcal{M} is a field of modular functions for K ; we do not know if it is in fact the field of all such functions.

The following identities can be proved by straight-forward manipulation of infinite products.

Lemma 3.1. *With $\omega = e^{2\pi i/3}$,*

$$\begin{aligned}\theta(p; p^2) &= p^{\frac{1}{12}} \phi_1, & \theta(-1; p^2) &= 2p^{-\frac{1}{6}} \phi_2, & \theta(-p; p^2) &= p^{\frac{1}{12}} \frac{1}{\phi_1 \phi_2}, \\ \theta(p\omega; p^2) &= p^{\frac{1}{12}} \frac{\phi_3}{\phi_5}, & \theta(-\omega; p^2) &= -\omega^2 p^{-\frac{1}{6}} \frac{\phi_4}{\phi_5}, & \theta(\omega; p^2) &= (1 - \omega) p^{-\frac{1}{6}} \phi_5, \\ \theta(-p\omega; p^2) &= p^{\frac{1}{12}} \frac{\phi_5^2}{\phi_3 \phi_4}.\end{aligned}$$

The normalizer of K in $\text{PSL}(2, \mathbb{Z})$ is $\Gamma_0(3)$, consisting of transformations (3.2) with $c \equiv 0 \pmod{3}$. It is generated by $T(\tau) = \tau + 1$ and $U(\tau) = (\tau - 1)/(3\tau - 2)$, subject to the single relation $U^3 = 1$ [Ra].

Lemma 3.2. *The generators of $\Gamma_0(3)$ act on the functions $[k/2]$, $k \mid 12$, according to the following table, where $X = \sqrt{-i(3\tau - 2)}$.*

	$[1/2]$	$[1]$	$[3/2]$	$[2]$	$[3]$	$[6]$
T	$e^{\frac{\pi i}{24}} \frac{[1^3]}{[1/2, 2]}$	$e^{\frac{\pi i}{12}} [1]$	$e^{\frac{\pi i}{8}} \frac{[3^3]}{[3/2, 6]}$	$e^{\frac{\pi i}{6}} [2]$	$e^{\frac{\pi i}{4}} [3]$	$e^{\frac{\pi i}{2}} [6]$
U	$e^{-\frac{7\pi i}{24}} \frac{X[1^3]}{[1/2, 2]}$	$e^{-\frac{\pi i}{12}} X[1]$	$e^{-\frac{\pi i}{24}} \frac{X[3^3]}{[3/2, 6]}$	$e^{\frac{\pi i}{12}} \frac{X}{\sqrt{2}} [1/2]$	$e^{-\frac{\pi i}{12}} X[3]$	$e^{\frac{\pi i}{12}} \frac{X}{\sqrt{2}} [3/2]$

Proof. This is straight-forward to verify using (3.7). For instance,

$$\begin{aligned}[3/2](U\tau) &= \eta \left(\frac{3(\tau - 1)}{2(3\tau - 2)} \right) = \eta \left(\frac{1}{2} - \frac{1}{6\tau - 4} \right) = e^{\frac{\pi i}{24}} \frac{\eta \left(-\frac{1}{3\tau - 2} \right)^3}{\eta \left(-\frac{1}{6\tau - 4} \right) \eta \left(-\frac{2}{3\tau - 2} \right)} \\ &= e^{\frac{\pi i}{24}} X \frac{\eta(3\tau - 2)^3}{\eta(6\tau - 4) \eta \left(\frac{3\tau - 2}{2} \right)} = e^{-\frac{\pi i}{24}} X \frac{\eta(3\tau)^3}{\eta(6\tau) \eta \left(\frac{3\tau}{2} \right)}.\end{aligned}$$

□

Corollary 3.3. *The group $\Gamma_0(3)$ acts on \mathcal{M} according to the following table.*

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
T	$e^{-\frac{\pi i}{12}} \frac{1}{\phi_1 \phi_2}$	$e^{\frac{\pi i}{6}} \phi_2$	$e^{\frac{\pi i}{12}} \frac{\phi_5^3}{\phi_3 \phi_4}$	$e^{\frac{\pi i}{3}} \phi_4$	$e^{\frac{\pi i}{6}} \phi_5$
U	$e^{-\frac{5\pi i}{12}} \frac{1}{\phi_1 \phi_2}$	$e^{\frac{\pi i}{3}} \frac{\phi_1}{2}$	$e^{\frac{\pi i}{4}} \frac{\phi_5^3}{\phi_3 \phi_4}$	ϕ_3	ϕ_4

One may check that the action described in Corollary 3.3 factors to a faithful action of $\Gamma_0(3)/K$, which is a group of order 12^3 . We will only work with the subgroup of $\Gamma_0(3)$ generated by $t_1 = (UT^3)^3$ and $t_3 = (T^3U)^3$. We use the same notation as for Bäcklund transformations in view of Proposition 3.7 below.

Corollary 3.4. *The transformations t_1 and t_3 act on \mathcal{M} according to the following table, where we also introduce an auxiliary field automorphism σ .*

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
t_1	ϕ_1	$e^{\frac{3\pi i}{4}} \frac{1}{2\phi_1\phi_2}$	$e^{\frac{3\pi i}{2}} \phi_3$	$e^{\frac{5\pi i}{4}} \frac{\phi_5^3}{\phi_3\phi_4}$	$e^{\frac{3\pi i}{2}} \phi_5$
t_3	$2e^{\frac{\pi i}{2}} \phi_2$	$e^{\frac{3\pi i}{2}} \frac{\phi_1}{2}$	$-\phi_4$	ϕ_3	$e^{\frac{3\pi i}{2}} \phi_5$
σ	ϕ_1	ϕ_2	$-\phi_3$	$-\phi_4$	$-\phi_5$

These automorphisms satisfy (2.12) and thus generate the group Dic_3 .

Combining [R2, Lemma 9.1] and Lemma 3.1 gives the following relations between the generators of \mathcal{M} and the Hauptmodul ζ .

Lemma 3.5. *We have*

$$\begin{aligned} \zeta &= -2 \frac{\phi_1 \phi_2^2 \phi_5^3}{\phi_3 \phi_4^2}, & \zeta + 1 &= -\frac{\phi_1^2 \phi_2 \phi_5^3}{\phi_3^2 \phi_4}, & \zeta - 1 &= -3 \frac{\phi_1^2 \phi_2 \phi_3^2 \phi_5}{\phi_4}, \\ \zeta + 2 &= 6 \frac{\phi_1 \phi_2^2 \phi_4^2 \phi_5}{\phi_3}, & 2\zeta + 1 &= -3 \frac{\phi_5^{10}}{\phi_3^4 \phi_4^4}. \end{aligned}$$

In particular, the field of modular functions $\mathbb{C}(\zeta)$ for $\Gamma_0(6, 2)$ is a subfield of \mathcal{M} . Moreover,

$$\begin{aligned} \phi_1^{12} &= \frac{2^4(\zeta - 1)^2(\zeta + 1)^6}{\zeta^3(\zeta + 2)(2\zeta + 1)}, & \phi_2^{12} &= -\frac{\zeta^6(\zeta + 2)^2}{2^8(\zeta + 1)^3(\zeta - 1)(2\zeta + 1)}, \\ \phi_3^{12} &= \frac{(\zeta - 1)^4(\zeta + 2)(2\zeta + 1)}{3^6\zeta(\zeta + 1)^4}, & \phi_4^{12} &= -\frac{(\zeta - 1)(\zeta + 2)^4(2\zeta + 1)}{3^6\zeta^4(\zeta + 1)}, \\ \phi_5^6 &= \frac{(\zeta - 1)(\zeta + 2)(2\zeta + 1)}{3^3\zeta(\zeta + 1)}. \end{aligned}$$

It follows that

$$t_1(\zeta) = \frac{1}{\zeta}, \quad t_3(\zeta) = -\zeta - 1, \quad \sigma(\zeta) = \zeta.$$

In agreement with [R3, §2.9], these maps generate an action of $\text{Dic}_3 / \{\sigma = 1\} \simeq S_3$ on $\mathbb{C}(\zeta)$.

Corollary 3.6. *The field \mathcal{M} is closed under the differentiation (3.5).*

Proof. By Lemma 3.5, $\phi_j^{12} \in \mathbb{C}(\zeta)$ for each j . Thus,

$$\delta(\phi_j) = \frac{\phi_j}{12} \cdot \frac{\delta(\phi_j^{12})}{\phi_j^{12}} \in \mathcal{M}.$$

□

We are now ready to incorporate tau functions in our seed solution.

Proposition 3.7. *The following equations define an extension of \mathbf{X} to a field automorphism $\mathcal{F} \rightarrow \mathcal{M}$:*

$$\begin{aligned} \mathbf{X}(u) &= \frac{\phi_1^2 \phi_3^4}{2^{2/3} \phi_5^4}, & \mathbf{X}(v) &= -\frac{2^{4/3} \phi_2^2 \phi_4^4}{\phi_5^4}, \\ \mathbf{X}(\tau_0) &= \frac{1}{\phi_5}, & \mathbf{X}(\tau_1) &= -\frac{\phi_3 \phi_4}{\phi_5^2}, & \mathbf{X}(\tau_2) &= \frac{2^{-2/3} i \phi_5^4}{\phi_1^2 \phi_2^2 \phi_3^2 \phi_4^2}, \\ \mathbf{X}(\tau_3) &= \frac{e^{\frac{\pi i}{4}} \phi_5}{\phi_3}, & \mathbf{X}(\tau_4) &= \frac{e^{\frac{3\pi i}{4}} \phi_5}{\phi_4}. \end{aligned}$$

This extension satisfies (3.6), as well as the identities

$$\mathbf{X} \circ t_j = t_j \circ \mathbf{X}, \quad j = 1, 3, \quad (3.8a)$$

$$\mathbf{X} \circ s_j = \mathbf{X}, \quad j = 0, 1, 3, 4. \quad (3.8b)$$

It is useful to note that

$$\mathbf{X}(u) \simeq \frac{(\zeta + 1)^{1/3} (\zeta - 1)}{\zeta^{1/6} (\zeta + 2)^{1/2} (2\zeta + 1)^{1/2}}, \quad (3.9a)$$

$$\mathbf{X}(v) \simeq \frac{\zeta^{1/3} (\zeta + 2)}{(\zeta + 1)^{1/6} (\zeta - 1)^{1/2} (2\zeta + 1)^{1/2}}, \quad (3.9b)$$

$$\mathbf{X}(\tau_0) \simeq \frac{\zeta^{1/6} (\zeta + 1)^{1/6}}{(2\zeta + 1)^{1/6} (\zeta - 1)^{1/6} (\zeta + 2)^{1/6}}, \quad (3.9c)$$

$$\mathbf{X}(\tau_1) \simeq \frac{(\zeta - 1)^{1/12} (\zeta + 2)^{1/12}}{(2\zeta + 1)^{1/6} (\zeta + 1)^{1/12} \zeta^{1/12}}, \quad (3.9d)$$

$$\mathbf{X}(\tau_2) \simeq \frac{(2\zeta + 1)^{2/3}}{\zeta^{1/3} (\zeta + 1)^{1/3} (\zeta - 1)^{1/3} (\zeta + 2)^{1/3}}, \quad (3.9e)$$

$$\mathbf{X}(\tau_3) \simeq \frac{(2\zeta + 1)^{1/12} (\zeta + 1)^{1/6} (\zeta + 2)^{1/12}}{\zeta^{1/12} (\zeta - 1)^{1/6}}, \quad (3.9f)$$

$$\mathbf{X}(\tau_4) \simeq \frac{(2\zeta + 1)^{1/12} \zeta^{1/6} (\zeta - 1)^{1/12}}{(\zeta + 1)^{1/12} (\zeta + 2)^{1/6}}, \quad (3.9g)$$

where $f \simeq g^{1/N}$ (with $f, g \in \mathcal{F}$) means that $f^N/g \in \mathbb{C}$. We will also need the identities

$$\mathbf{X}\left(\frac{\delta(u)}{u}\right) = \frac{\zeta^4 + 14\zeta^3 + 24\zeta^2 + 14\zeta + 1}{6(2\zeta + 1)^3}, \quad (3.10a)$$

$$\mathbf{X}\left(\frac{\delta(v)}{v}\right) = \frac{\zeta^4 - 10\zeta^3 - 12\zeta^2 - 4\zeta - 2}{6(2\zeta + 1)^3}, \quad (3.10b)$$

$$\mathbf{X}\left(\frac{\delta(\tau_0)}{\tau_0}\right) = -\frac{(\zeta^2 + \zeta + 1)^2}{6(2\zeta + 1)^3}, \quad (3.10c)$$

$$\mathbf{X} \left(\frac{\delta(\tau_1)}{\tau_1} \right) = -\frac{2\zeta^4 + 4\zeta^3 - 6\zeta^2 - 8\zeta - 1}{12(2\zeta + 1)^3}, \quad (3.10d)$$

$$\mathbf{X} \left(\frac{\delta(\tau_2)}{\tau_2} \right) = -\frac{(\zeta^2 + \zeta + 1)(2\zeta^2 + 2\zeta - 1)}{3(2\zeta + 1)^3}, \quad (3.10e)$$

$$\mathbf{X} \left(\frac{\delta(\tau_3)}{\tau_3} \right) = \frac{\zeta^4 - 4\zeta^3 - 12\zeta^2 - 4\zeta + 1}{12(2\zeta + 1)^3}, \quad (3.10f)$$

$$\mathbf{X} \left(\frac{\delta(\tau_4)}{\tau_4} \right) = \frac{\zeta^4 + 8\zeta^3 + 6\zeta^2 - 4\zeta - 2}{12(2\zeta + 1)^3}. \quad (3.10g)$$

Proof of Proposition 3.7. To prove that \mathbf{X} is well-defined, we only need to check that it respects the relations (2.11). This is clear since, by Lemma 3.5,

$$t = 16 \frac{\phi_1^4 \phi_2^8 \phi_3^8 \phi_4^{16}}{\phi_5^{24}}, \quad 1 - t = -\frac{(\zeta + 1)(\zeta - 1)^3}{(2\zeta + 1)^3} = \frac{\phi_1^8 \phi_2^4 \phi_3^{16} \phi_4^8}{\phi_5^{24}}.$$

We must also check that (3.6) holds on each of the generators u, v and τ_j . Let g be one of these generators. By (3.9), $\mathbf{X}(g^{12}) \in \mathbb{C}(\zeta)$. Thus, we can rewrite the relevant identity as

$$\mathbf{X} \left(\frac{\delta(g)}{g} \right) = \frac{\delta(\mathbf{X}(g^{12}))}{12\mathbf{X}(g^{12})},$$

which is elementary to verify using (3.5) and (3.10). The remaining statements are easy to verify. \square

In particular, $\mathbf{X}(s_j(\tau_j)/\tau_j) = 1$ for $j = 0, 1, 3, 4$, that is,

$$\begin{aligned} \frac{i(2\zeta + 1)^3}{3\zeta(\zeta + 1)(\zeta - 1)(\zeta + 2)} \mathbf{X} \left(\frac{u^2 v^2 \tau_0^2}{\tau_2} \right) &= -i \mathbf{X} \left(\frac{\tau_1^2}{uv\tau_2} \right) \\ &= \frac{(2\zeta + 1)}{(\zeta + 1)(1 - \zeta)} \mathbf{X} \left(\frac{u\tau_3^2}{\tau_2} \right) = \frac{2\zeta + 1}{\zeta(\zeta + 2)} \mathbf{X} \left(\frac{v\tau_4^2}{\tau_2} \right) = 1. \end{aligned} \quad (3.11)$$

4. IDENTIFICATION OF TAU FUNCTIONS

Let k_0, \dots, k_3 be integers such that $\sum_j k_j$ is even and write

$$n = \frac{k_0 + k_1 + k_2 + k_3}{2}.$$

As in [R4], we write $t^{(k_0, k_1, k_2, k_3)} = T_n^{(k_0, k_1, k_2, k_3)}$, where $T_n^{(k_0, k_1, k_2, k_3)}$ was introduced in [R3]. In general, $T_n^{(k_0, k_1, k_2, k_3)}$ is a function of $m = 2n - \sum_j k_j$ variables and one parameter ζ ; in this paper we are only concerned with the case $m = 0$. Our main result is that, up to elementary factors, the tau function $\mathbf{X}(\tau_{l_1, l_2, l_3, l_4})$ can be identified with $t^{(k_0, k_1, k_2, k_3)}$, where

$$k_0 = l_0 + l_2 + l_3, \quad k_1 = -l_2, \quad k_2 = l_0 + l_2 + l_4, \quad k_3 = l_0 + l_1 + l_2, \quad (4.1a)$$

with l_0 given by (2.21). Equivalently,

$$l_0 = n, \quad l_1 = k_1 + k_3 - n, \quad l_2 = -k_1, \quad (4.1b)$$

$$l_3 = k_0 + k_1 - n, \quad l_4 = k_1 + k_2 - n. \quad (4.1c)$$

The map $(l_1, l_2, l_3, l_4) \mapsto (k_0, k_1, k_2, k_3)$ is a bijection from \mathbb{Z}^4 to the sublattice of \mathbb{Z}^4 defined by $k_0 + k_1 + k_2 + k_3 \in 2\mathbb{Z}$.

Note also that, by (2.6),

$$s_2 s_0 T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} (s_2 s_0)^{-1} = T_1^{\hat{l}_1} T_2^{\hat{l}_2} T_3^{\hat{l}_3} T_4^{\hat{l}_4},$$

where $\hat{l}_j = l_j + l_0 + l_2$ for $j \neq 2$ and $\hat{l}_2 = l_1 + l_2 + l_3 + l_4$. We can then write (4.1a) more symmetrically as $k_{0,1,2,3} = \hat{l}_2 + \hat{l}_{3,0,4,1}$.

We summarize the main facts about the function $t^{(k_0, k_1, k_2, k_3)}$ as follows, see [R3, Cor. 3.9 and Thm. 4.1], [R4, Thm. 4.1]. For $a \in \mathbb{C} \cup \{\infty\}$, we write $\text{o}_a(f)$ for the order of a meromorphic function f at the point a , that is, $\lim_{z \rightarrow a} (z - a)^{-\text{o}_a(f)} f(z)$ and $\lim_{z \rightarrow \infty} z^{\text{o}_\infty(f)} f(z)$ are finite and non-zero. We will write (cf. (3.4))

$$\Lambda = \{0, 1, -1, -2, -1/2\}. \quad (4.2)$$

Proposition 4.1 ([R3, R4]). *The functions $t^{(\mathbf{k})} = t^{(k_0, k_1, k_2, k_3)}(\zeta)$ is a rational function in ζ with no poles outside Λ . At $\zeta = 0$ and $\zeta = -2$, it has order*

$$\text{o}_0(t^{(\mathbf{k})}) = (k_1 + k_2)(2n - k_1 - k_2 - 1) + \max((n + 1)(k_1 + k_2 - n), 0), \quad (4.3a)$$

$$\text{o}_{-2}(t^{(\mathbf{k})}) = \left\lfloor \frac{(k_1 + k_2 - 1)^2}{4} \right\rfloor - (k_1 + k_2)(n - 1). \quad (4.3b)$$

Moreover, the functions $t^{(\mathbf{k})}$ are uniquely determined by the two recursions

$$\begin{aligned} t^{(\mathbf{k}-2\mathbf{e}_0)} t^{(\mathbf{k}+\mathbf{e}_0+\mathbf{e}_1)} &= \zeta^2 (\zeta + 1)(\zeta - 1)(2\zeta + 1)^2 \\ &\times \left(\frac{1}{2k_0 - 1} t^{(\mathbf{k})} \frac{dt^{(\mathbf{k}-\mathbf{e}_0+\mathbf{e}_1)}}{d\zeta} - \frac{1}{2k_0 + 1} \frac{dt^{(\mathbf{k})}}{d\zeta} t^{(\mathbf{k}-\mathbf{e}_0+\mathbf{e}_1)} \right) \\ &+ \frac{\zeta(2\zeta + 1)}{2(2k_0 - 1)(2k_0 + 1)(\zeta + 2)} A^{(\mathbf{k})} t^{(\mathbf{k})} t^{(\mathbf{k}-\mathbf{e}_0+\mathbf{e}_1)}, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} t^{(\mathbf{k}-2\mathbf{e}_0)} t^{(\mathbf{k}+\mathbf{e}_0-\mathbf{e}_1)} &= \frac{(\zeta + 1)(\zeta - 1)(\zeta + 2)^2(2\zeta + 1)^2}{\zeta^2} \\ &\times \left(\frac{1}{2k_0 - 1} t^{(\mathbf{k})} \frac{dt^{(\mathbf{k}-\mathbf{e}_0-\mathbf{e}_1)}}{d\zeta} - \frac{1}{2k_0 + 1} \frac{dt^{(\mathbf{k})}}{d\zeta} t^{(\mathbf{k}-\mathbf{e}_0-\mathbf{e}_1)} \right) \\ &+ \frac{(2\zeta + 1)(\zeta + 2)}{2(2k_0 - 1)(2k_0 + 1)\zeta^3} B^{(\mathbf{k})} t^{(\mathbf{k})} t^{(\mathbf{k}-\mathbf{e}_0-\mathbf{e}_1)}, \end{aligned} \quad (4.4b)$$

where \mathbf{e}_j are unit vectors and $A^{(\mathbf{k})}$ and $B^{(\mathbf{k})}$ certain explicit polynomials (see [R4]) in ζ and k_j , the three initial values

$$t^{(0,0,0,0)} = t^{(1,-1,0,0)} = 1, \quad (4.4c)$$

$$t^{(0,-1,-1,0)} = -\frac{2\zeta^2(\zeta-1)(\zeta+1)^2(2\zeta+1)}{(\zeta+2)^2} \quad (4.4d)$$

and the four symmetries

$$t^{(k_0,k_1,k_2,k_3)}(\zeta) = (2\zeta+1)^{(k_0+k_2-n)(n-1)} \left(\frac{\zeta}{\zeta+2} \right)^{(k_1+k_2-n)(n-1)} t^{(k_1,k_0,k_3,k_2)}(\zeta) \quad (4.4e)$$

$$\begin{aligned} &= \left(-\frac{1}{12} \right)^{n+1} \frac{1}{\zeta^{2(k_1+k_2+2n+3)}(\zeta+1)^{2(k_0+k_1+2n+3)}(\zeta-1)^{2(k_2+k_3+1)}} \\ &\quad \times \frac{(\zeta+2)^{2(k_1+k_2+n+2)}}{(2\zeta+1)^{2(k_0+k_2+1)}} t^{(-k_0-1,-k_1-1,-k_2-1,-k_3-1)}(\zeta) \end{aligned} \quad (4.4f)$$

$$= \left(\frac{\zeta^3(2\zeta+1)}{\zeta+2} \right)^{n(n-1)} t^{(k_0,k_1,k_3,k_2)}(\zeta^{-1}) \quad (4.4g)$$

$$= \left(\frac{\zeta-1}{\zeta+2} \right)^{n(n-1)} t^{(k_2,k_1,k_0,k_3)}(-\zeta-1). \quad (4.4h)$$

We have only given the order at the cusps corresponding to the singular point $t = 0$ of (2.1). The behaviour at the other cusps follows using (4.4g)–(4.4h).

Let us introduce the normalizing factor $\phi_{l_1 l_2 l_3 l_4} \in \mathcal{F}$ given by

$$\begin{aligned} \phi_{l_1 l_2 l_3 l_4} &= \frac{(-1)^{\binom{l_1+1}{3} + \binom{l_3+1}{3} + \binom{l_4+1}{3} + \binom{l_2+1}{2} + l_1 l_3 + l_2} l_4! \binom{l_3+1}{2} + \binom{l_4+1}{2} - l_1^2 l_3 + l_1 l_4^2 + l_2 + l_3 + l_4}{2^{l_0(l_0-1) + l_1^2 + l_3^2 + l_4^2}} \\ &\quad \times \zeta^{l_4^2 - l_0(l_0-1) - (l_0+l_2)(l_2+l_4)} (\zeta+1)^{l_3^2 - l_0(l_0-1) - (l_0+l_2)(l_2+l_3)} \\ &\quad \times (\zeta-1)^{(l_0+l_2)(l_1+l_4) - (l_2+l_3)^2 - l_3} (\zeta+2)^{-3l_2(l_0+l_2+l_4) - (l_0+l_4)(l_4+1)} \\ &\quad \times (2\zeta+1)^{-l_0^2 - l_1(l_0+l_1+3l_2+1) - l_2} u^{\frac{1}{2}(l_1-l_3)(l_1+l_3+2l_4-1) + 2l_2(l_0+l_2)} \\ &\quad \times v^{\frac{1}{2}(l_1-l_4)(l_1+l_4+2l_3-1) + 2l_2(l_0+l_2)} \tau_0^{l_0+1} \tau_1^{l_1} \tau_2^{l_2} \tau_3^{l_3} \tau_4^{l_4}. \end{aligned}$$

It is sometimes useful to write, with notation as in (3.9),

$$\begin{aligned} \phi_{l_1 l_2 l_3 l_4} &\simeq \zeta^{\frac{1}{12}} (-10l_0^2 - l_1^2 - l_3^2 + 14l_4^2 - 6l_0 l_4 + 16l_0 + 6l_4 + 2) \\ &\quad \times (\zeta+1)^{\frac{1}{12}} (-10l_0^2 - l_1^2 + 14l_3^2 - l_4^2 - 6l_0 l_3 + 16l_0 + 6l_3 + 2) \\ &\quad \times (\zeta-1)^{\frac{1}{12}} (-6l_0^2 - 3l_1^2 - 6l_3^2 - 3l_4^2 + 6l_0 l_3 - 6l_3 - 2) \\ &\quad \times (\zeta+2)^{\frac{1}{12}} (6l_0^2 - 3l_1^2 - 3l_3^2 - 6l_4^2 + 6l_0 l_4 - 12l_0 - 6l_4 - 2) \\ &\quad \times (2\zeta+1)^{\frac{1}{12}} (-6l_0^2 - 6l_1^2 - 3l_3^2 - 3l_4^2 + 6l_0 l_1 - 6l_1 - 2). \end{aligned} \quad (4.5)$$

Moreover, one can compute

$$\begin{aligned} \frac{\delta(\phi_{l_1 l_2 l_3 l_4})}{\phi_{l_1 l_2 l_3 l_4}} = \frac{1}{12(2\zeta + 1)^3} & \left(-(26\zeta^4 + 70\zeta^3 + 6\zeta^2 - 38\zeta - 10)l_0^2 \right. \\ & - (14\zeta^4 + 28\zeta^3 - 14\zeta - 1)l_1^2 + (\zeta^4 - 10\zeta^3 - 36\zeta^2 - 10\zeta + 1)l_3^2 \\ & + (\zeta^4 + 14\zeta^3 - 28\zeta - 14)l_4^2 + 6\zeta(\zeta + 1)(\zeta - 1)(\zeta + 2)l_1(l_0 - 1) \\ & + 6\zeta(\zeta + 2)(2\zeta + 1)l_3(l_0 - 1) - 6(\zeta + 1)(\zeta - 1)(2\zeta + 1)l_4(l_0 - 1) \\ & \left. + 2(\zeta - 1)(2\zeta + 1)(5\zeta^2 + 17\zeta + 8)l_0 - 2(\zeta^2 + \zeta + 1)^2 \right). \end{aligned} \quad (4.6)$$

Let $(Y_k)_{k \in \mathbb{Z}}$ be the solution to the recursion

$$Y_{k+1}Y_{k-1} = 2(2k+1)Y_k^2, \quad Y_0 = Y_1 = 1,$$

that is,

$$Y_k = \begin{cases} \prod_{j=1}^k \frac{(2j-1)!}{(j-1)!}, & k \geq 0, \\ \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{2k+1}} \prod_{j=1}^{-k-1} \frac{(2j-1)!}{(j-1)!}, & k < 0. \end{cases}$$

We can then formulate our main result as follows.

Theorem 4.2. *We have*

$$\mathbf{X} \left(\frac{\tau_{l_1 l_2 l_3 l_4}}{\phi_{l_1 l_2 l_3 l_4}} \right) = Y_{k_0} Y_{k_1} Y_{k_2} Y_{k_3} t^{(k_0, k_1, k_2, k_3)}, \quad (4.7)$$

where k_j and l_j are related by (4.1).

After our preparations (which take up a large portion also of the papers [R3, R4]) the proof of Theorem 4.2 is straight-forward.

Proof. It is enough to show that, if we use (4.7) to define $t^{(\mathbf{k})}$, all the properties (4.4) are valid. We first apply \mathbf{X} to (2.24) and substitute (4.7). Using that

$$\mathbf{X} \left(u^2 v^2 \frac{\phi_{l_1, l_2-1, l_3+1, l_4} \phi_{l_1+1, l_2, l_3-1, l_4+1}}{\phi_{l_1, l_2, l_3, l_4} \phi_{l_1+1, l_2-1, l_3, l_4+1}} \right) = -\frac{\zeta + 2}{16\zeta(2\zeta + 1)^4}$$

and

$$\frac{Y_{k+1}Y_{k-2}}{Y_kY_{k-1}} = 4(2k+1)(2k-1),$$

we find that (4.4a) holds with

$$\begin{aligned} A^{(\mathbf{k})} = -8(2\zeta + 1)^3 & \left(R(l_0, l_1, l_3, l_4) + \left(k_0 - \frac{1}{2} \right) \frac{\delta(\phi_{l_1 l_2 l_3 l_4})}{\phi_{l_1 l_2 l_3 l_4}} \right. \\ & \left. - \left(k_0 + \frac{1}{2} \right) \frac{\delta(\phi_{l_1+1, l_2-1, l_3, l_4+1})}{\phi_{l_1+1, l_2-1, l_3, l_4+1}} \right). \end{aligned}$$

Similarly, (2.23) yields (4.4b) with

$$B^{(\mathbf{k})} = -8(2\zeta + 1)^3 \left(Q(l_0, l_1, l_3, l_4) + \left(k_0 - \frac{1}{2} \right) \frac{\delta(\phi_{l_1 l_2 l_3 l_4})}{\phi_{l_1 l_2 l_3 l_4}} - \left(k_0 + \frac{1}{2} \right) \frac{\delta(\phi_{l_1, l_2+1, l_3-1, l_4})}{\phi_{l_1, l_2+1, l_3-1, l_4}} \right).$$

Using (4.6), one may check that this agrees with the explicit expressions for $A^{(\mathbf{k})}$ and $B^{(\mathbf{k})}$ given in [R4].

The initial values are trivial to check. To prove (4.4e), we apply the identity $\mathbf{X} \circ s_1 s_4 = \mathbf{X}$, which follows from (3.8b), to $\tau_{l_1 l_2 l_3 l_4}$. Since, by (2.6b),

$$s_1 s_4 T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} = T_1^{-l_1} T_2^{l_1+l_2+l_4} T_3^{l_3} T_4^{-l_4} s_1 s_4,$$

we obtain $\mathbf{X}(\tau_{l_1 l_2 l_3 l_4}) = \mathbf{X}(\tau_{-l_1, l_1+l_2+l_4, l_3, -l_4})$. We substitute (4.7) and write

$$\frac{\phi_{l_1 l_2 l_3 l_4}}{\phi_{-l_1, l_1+l_2+l_4, l_3, -l_4}} = \frac{(\zeta + 2)^{l_0 l_4 - 2l_4} (2\zeta + 1)^{l_0 l_1 - l_1 + l_4}}{\zeta^{l_0 l_4}} \left(\frac{\tau_1^2}{i u v \tau_2} \right)^{l_1} \left(\frac{v \tau_4^2}{\tau_2} \right)^{l_4}.$$

Applying (3.11) yields (4.4e) after simplification.

Similarly, starting from $\mathbf{X} \circ s_0 s_1 s_3 s_4 = \mathbf{X}$ and using $s_0(\tau_0) = T_2(\tau_0)$, we get $\mathbf{X}(\tau_{l_1 l_2 l_3 l_4}) = \mathbf{X}(\tau_{-l_1, 1-l_2, -l_3, -l_4})$. Writing

$$\begin{aligned} \frac{\phi_{l_1 l_2 l_3 l_4}}{\phi_{-l_1, 1-l_2, -l_3, -l_4}} &= (-1)^{l_1 l_3 + l_1 l_4 + l_3 l_4} \left(\frac{2^6 \zeta^5 (\zeta + 1)^5 (\zeta - 1) (2\zeta + 1)^5 u^2 v^2 \tau_0^2}{i (\zeta + 2)^5 \tau_2} \right)^{l_0+1} \\ &\times \left(\frac{\tau_1^2}{i (2\zeta + 1)^2 u v \tau_2} \right)^{l_1} \left(-\frac{(\zeta + 1) (2\zeta + 1) u \tau_3^2}{(\zeta - 1)^3 \tau_2} \right)^{l_3} \left(\frac{\zeta (2\zeta + 1) v \tau_4^2}{(\zeta + 2)^3 \tau_2} \right)^{l_4} \end{aligned}$$

and noting that

$$\prod_{j=0}^3 \frac{Y_{k_j}}{Y_{-k_j-1}} = \prod_{j=0}^3 \frac{(-1)^{\frac{k_j(k_j+1)}{2}}}{2^{2k_j+1}} = \frac{(-1)^{l_1 l_3 + l_1 l_4 + l_3 l_4}}{16^{l_0+1}}$$

gives (4.4f).

Next, by Corollary 2.2 and (3.8a),

$$t_1 \mathbf{X}(\tau_{l_1 l_2 l_3 l_4}) = i^{1+(2l_2-1)(l_1+l_3+l_4)+(l_3-l_1)(l_4-l_1)(l_4-l_3)} \mathbf{X}(\tau_{l_4 l_2 l_3 l_1}). \quad (4.8)$$

Then, (4.4g) follows from the easily verified identity

$$\begin{aligned} \frac{t_1(\phi_{l_1 l_2 l_3 l_4})}{\phi_{l_4 l_2 l_3 l_1}} &= i^{1+(2l_2-1)(l_1+l_3+l_4)+(l_3-l_1)(l_4-l_1)(l_4-l_3)} \\ &\times \left(\frac{\zeta^3 (2\zeta + 1)}{\zeta + 2} \right)^{l_0(l_0-1)} \left(\frac{(2\zeta + 1)q}{\zeta(\zeta + 2)} \right)^{l_2}. \end{aligned}$$

The last symmetry, (4.4h), is proved similarly. \square

5. APPLICATIONS

5.1. Behaviour at singular points. As a first application of Theorem 4.2, we can compute the leading behaviour of the tau functions at the cusps. Let

$$C_0(l_0, l_1, l_3, l_4) = \frac{l_0^2}{6} - \frac{l_1^2}{12} - \frac{l_3^2}{12} + \frac{l_4^2}{6} - \frac{l_0 l_4}{2} + \frac{l_0}{3} - \frac{l_4}{2} + \frac{1}{6} + \max((l_0 + 1)l_4, 0),$$

$$C_{-2}(l_0, l_1, l_3, l_4) = -\frac{l_0^2}{2} - \frac{l_1^2}{4} - \frac{l_3^2}{4} - \frac{l_4^2}{2} - \frac{l_0 l_4}{2} + \frac{l_4}{2} - \frac{1}{6} + \left\lceil \frac{(l_0 + l_4 - 1)^2}{4} \right\rceil$$

and define

$$C_{-1}(l_0, l_1, l_3, l_4) = C_0(l_0, l_1, l_4, l_3), \quad C_\infty(l_0, l_1, l_3, l_4) = C_0(l_0, l_4, l_3, l_1),$$

$$C_1(l_0, l_1, l_3, l_4) = C_{-2}(l_0, l_1, l_4, l_3), \quad C_{-1/2}(l_0, l_1, l_3, l_4) = C_{-2}(l_0, l_4, l_3, l_1).$$

Corollary 5.1. *With Λ as in (4.2) and using the notation \simeq as in (3.9),*

$$\mathbf{X}(\tau_{l_1 l_2 l_3 l_4}) \simeq \prod_{a \in \Lambda} (\zeta - a)^{C_a(l_0, l_1, l_3, l_4)} p(\zeta), \quad (5.1)$$

where p is a polynomial of degree

$$\deg(p) = - \sum_{a \in \Lambda \cup \{\infty\}} C_a(l_0, l_1, l_3, l_4)$$

$$= 2 \binom{l_2}{2} + \sum_{j \in \{1, 3, 4\}} \left(2 \binom{l_2 + l_j + 1}{2} + \max((l_0 + 1)l_j, 0) + \left\lceil \frac{(l_0 + l_j - 1)^2}{4} \right\rceil \right),$$

which does not vanish at Λ .

Proof. Combining (4.3), (4.5) and (4.7) yields the given expressions for C_0 and C_{-2} . By (4.8) and the corresponding equation for t_3 , it follows that $\mathbf{o}_a(\mathbf{X}(\tau_{l_1 l_2 l_3 l_4})) = C_a(l_0, l_1, l_3, l_4)$ for each $a \in \Lambda \cup \infty$. This proves (5.1) and the first expression for $\deg(p)$. The second expression follows by a direct computation. \square

Having understood the behaviour of the tau functions at the cusps, it is easy to understand the corresponding solutions. Let

$$q_{l_1 l_2 l_3 l_4} = \mathbf{X}(T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} q) \in \mathbb{C}(\zeta).$$

Recall that $q = q_{l_1 l_2 l_3 l_4}$ solves (2.1), with t given by (3.3) and

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{l_1^2}{2}, -\frac{l_4^2}{2}, \frac{l_3^2}{2}, \frac{1 - l_0^2}{2} \right).$$

Corollary 5.2. *Define $\chi(k)$ as 1 for k odd and 0 for k even. Then,*

$$q_{l_1 l_2 l_3 l_4} = \frac{\zeta^{1+|l_0|\delta_{l_4,0}} (\zeta + 2)^{1+\chi(l_1+l_3)}}{(2\zeta + 1)^{1+\chi(l_3+l_4)}} f(\zeta) \quad (5.2a)$$

$$= 1 + \frac{(\zeta + 1)^{1+|l_0|\delta_{l_3,0}} (\zeta - 1)^{1+\chi(l_1+l_4)}}{(2\zeta + 1)^{1+\chi(l_3+l_4)}} g(\zeta), \quad (5.2b)$$

with f and g rational functions with no zeroes or poles in Λ . Moreover,

$$o_\infty(q_{l_1 l_2 l_3 l_4}) = 1 + |l_0| \delta_{l_1, 0}. \quad (5.3)$$

Proof. If we substitute (5.1) in (2.22) and simplify, using

$$\begin{aligned} \max(k(l+1), 0) + \max(k(l-1), 0) - 2 \max(kl, 0) &= |k| \delta_{l, 0}, \\ \left\lfloor \frac{(l+1)^2}{4} \right\rfloor + \left\lfloor \frac{(l-1)^2}{4} \right\rfloor - 2 \left\lfloor \frac{l^2}{4} \right\rfloor &= \chi(l), \end{aligned}$$

we obtain (5.2a) and (5.3). Applying t_3 , using (2.14) and (3.8a), yields (5.2b). \square

Corollary 5.2 immediately gives the behaviour of the solutions near the singular points of (2.1). For instance, near $t = 0$, (3.3) behaves either as $t \sim \zeta$ or as $t \sim (\zeta + 2)^3$. The first branch corresponds to $q \sim t^{1+|l_0|\delta_{l_4, 0}}$ and the second branch to $q \sim t^{1/3}$ or $q \sim t^{2/3}$, depending on the parity of $l_1 + l_3$. In the terminology of [R3, §2.9], the first type of solution appears at the hyperbolic cusps and the second type at the trigonometric cusps. In the context of the XYZ model, these cusps corresponds to degenerations to the XY and XXZ model, respectively.

5.2. Properties of the functions $t^{(\mathbf{k})}$. We can apply Theorem 4.2 to deduce new properties of the functions $t^{(\mathbf{k})}$. For instance, we can obtain the following new symmetry. We do not know how to obtain this result without using the relation to tau functions.

Corollary 5.3. *The functions $t^{(k_0, k_1, k_2, k_3)}$ satisfy*

$$\begin{aligned} t^{(k_0, k_1, k_2, k_3)}(\zeta) &= (-1)^{(k_0+k_1+n)(k_1+k_3+n)} \frac{Y_{n-k_0} Y_{n-k_1} Y_{n-k_2} Y_{n-k_3}}{Y_{k_0} Y_{k_1} Y_{k_2} Y_{k_3}} \\ &\quad \times \left(\frac{\zeta^{k_1+k_2-n} (\zeta+1)^{k_0+k_1-n}}{(\zeta-1)^{k_0+k_1-n} (\zeta+2)^{k_1+k_2-n} (2\zeta+1)^{k_1+k_3-n}} \right)^{n-1} \\ &\quad \times t^{(n-k_0, n-k_1, n-k_2, n-k_3)}(\zeta). \end{aligned} \quad (5.4)$$

Proof. Proceeding as in the proof of Theorem 4.2 but starting from the identity $\mathbf{X} \circ s_1 s_3 s_4 = \mathbf{X}$ gives $\mathbf{X}(\tau_{l_1 l_2 l_3 l_4}) = \mathbf{X}(\tau_{-l_1, -l_0 - l_2, -l_3, -l_4})$. Substituting (4.7) and writing

$$\frac{\phi_{l_1 l_2 l_3 l_4}}{\phi_{-l_1, -l_0 - l_2, -l_3, -l_4}} = (-1)^{l_1 l_3} \left(\frac{(\zeta-1)^{l_3} (\zeta+2)^{l_4} (2\zeta+1)^{l_1}}{\zeta^{l_4} (\zeta+1)^{l_3}} \right)^{l_0 - 1}$$

we obtain (5.4) after simplification. \square

The symmetries (4.4e)–(4.4h) and (5.4) generate the group $G = S_4 \times S_2 \times S_2$. This is the full set of symmetries arising from (3.8). Indeed, the group generated by s_0, s_1, s_3, s_4, t_1 and t_3 under the relations (2.5) is equal to G .

As another application, we can obtain further bilinear relations for $t^{(\mathbf{k})}$. Probably, any such relation can also be found using the method explained in [R4, §4] (see also [Z, §4.3]), that is, by combining minor relations for the determinant

defining $T_n^{(\mathbf{k})}$ with differential relations derived from [R4, Thm. 3.3]. However, the approach based on Bäcklund transformations is more systematic. There are many such relations, but we will only give one example.

Proposition 5.4. *The functions $t^{(\mathbf{k})} = t^{(k_0, k_1, k_2, k_3)}$ satisfy the bilinear relation*

$$-\frac{(2k_0+1)(2k_1+1)(\zeta+2)^2}{\zeta^2} t^{(\mathbf{k}+\mathbf{e}_0+\mathbf{e}_1)} t^{(\mathbf{k}-\mathbf{e}_0-\mathbf{e}_1)} \\ = A \left(\frac{d^2 t^{(\mathbf{k})}}{d\zeta^2} t^{(\mathbf{k})} - \left(\frac{dt^{(\mathbf{k})}}{d\zeta} \right)^2 \right) + B \frac{dt^{(\mathbf{k})}}{d\zeta} t^{(\mathbf{k})} + \frac{C}{4} (t^{(\mathbf{k})})^2, \quad (5.5)$$

where \mathbf{e}_j are unit vectors and

$$A = \zeta(\zeta+1)^2(\zeta-1)^2(\zeta+2)(2\zeta+1), \\ B = 2(\zeta+1)^2(\zeta-1)(\zeta^3-3\zeta^2-6\zeta-1),$$

$$C = (39\zeta^4 + 110\zeta^3 + 116\zeta^2 + 50\zeta + 9)k_0^2 + (35\zeta^4 + 110\zeta^3 + 124\zeta^2 + 50\zeta + 5)k_1^2 \\ + (31\zeta^4 + 70\zeta^3 + 32\zeta^2 - 14\zeta - 11)k_2^2 + (19\zeta^4 + 46\zeta^3 + 32\zeta^2 + 10\zeta + 1)k_3^2 \\ + 2(29\zeta^4 + 110\zeta^3 + 136\zeta^2 + 50\zeta - 1)k_0k_1 + 2(\zeta-1)(35\zeta^3 + 93\zeta^2 + 87\zeta + 25)k_0k_2 \\ + 2(\zeta-1)^2(5\zeta^2 + 8\zeta + 5)k_0k_3 + 2(\zeta-1)(9\zeta^3 + 19\zeta^2 + 17\zeta + 3)k_1k_2 \\ + 2(\zeta-1)(27\zeta^3 + 73\zeta^2 + 71\zeta + 21)k_1k_3 + 2(17\zeta^4 + 58\zeta^3 + 48\zeta^2 - 2\zeta - 13)k_2k_3 \\ - 2(3\zeta^4 - 52\zeta^3 - 136\zeta^2 - 112\zeta - 27)k_0 + 2(\zeta^4 + 52\zeta^3 + 128\zeta^2 + 112\zeta + 31)k_1 \\ - 2(27\zeta^4 + 68\zeta^3 + 44\zeta^2 - 16\zeta - 15)k_2 - 2(15\zeta^4 + 44\zeta^3 + 44\zeta^2 + 8\zeta - 3)k_3 \\ + 8(\zeta+1)^2(\zeta+2)(2\zeta+1).$$

Proof. Let ψ be the prefactor in (3.5), so that $\delta = \psi \cdot d/d\zeta$ on $\mathbb{C}(\zeta)$. Substituting (4.7) in (2.25), using

$$\frac{\phi_{l_1, l_2+1, l_3-1, l_4} \phi_{l_1, l_2-1, l_3+1, l_4}}{\phi_{l_1 l_2 l_3 l_4}^2} = \frac{(-1)^{l_3+l_4} \mathbf{i}}{16u\zeta^2(2\zeta+1)^2},$$

we find that (5.5) holds with

$$A = 4(\zeta+2)^2(2\zeta+1)^2 \frac{\psi^2}{t}, \quad B = 4(\zeta+2)^2(2\zeta+1)^2 \psi \left(\frac{1}{t} \frac{d\psi}{d\zeta} - 1 \right), \\ C = 16(\zeta+2)^2(2\zeta+1)^2 \left(S(l_0, l_1, l_3, l_4) + \frac{\psi}{t} \frac{d}{d\zeta} \left(\frac{\delta(\phi_{l_1 l_2 l_3 l_4})}{\phi_{l_1 l_2 l_3 l_4}} \right) - \frac{\delta(\phi_{l_1 l_2 l_3 l_4})}{\phi_{l_1 l_2 l_3 l_4}} \right).$$

Using (4.6), one may check that this agrees with the given expressions. \square

Proposition 5.4 settles some conjectures for polynomials related to solvable models. In [BM1], Bazhanov and Mangazeev found that the ground state eigenvalue for the Q -operator of a certain XYZ chain can be expressed in terms of special polynomials $\mathcal{P}_n(x, z)$. In [BM2], it was conjectured that, as a polynomial

in x , the highest and lowest coefficients of \mathcal{P}_n are Painlevé tau functions. In [R3, §5], we showed that those coefficients are essentially $t^{(n,n,0,0)}$ and $t^{(n,n,1,-1)}$. Thanks to Theorem 4.2, this interesting relation between Painlevé VI and the eight-vertex model is now rigorously established. We can then obtain the recursions of [BM2, Conj. 1(b)] as special cases of Proposition 5.4. For instance, substituting $\mathbf{k} = (n, n, 0, 0)$ in (5.5), we find that $t_n = t^{(n,n,0,0)}$ satisfies

$$-\frac{(2n+3)(2n+1)(\zeta+2)^2}{\zeta^2} t_{n+1} t_{n-1} = A(t_n t_n'' - (t_n')^2) + B t_n' t_n + D_n t_n^2, \quad (5.6)$$

with

$$D_n = (33\zeta^4 + 110\zeta^3 + 128\zeta^2 + 50\zeta + 3)n^2 - (\zeta^4 - 52\zeta^3 - 132\zeta^2 - 112\zeta - 29)n + 2(\zeta+1)^2(2\zeta+1)(\zeta+2).$$

In [R2], we showed that the partition function for the three-colour model with domain wall boundary conditions can be expressed in terms of certain polynomials p_n , which are essentially equal to $t^{(n+1,n,0,-1)}$ [R3, Eq. (5.5)]. We find from (5.5) that $t_n = t^{(n+1,n,0,-1)}$ satisfies (5.6) with

$$D_n = (33\zeta^4 + 110\zeta^3 + 128\zeta^2 + 50\zeta + 3)n^2 + (17\zeta^4 + 140\zeta^3 + 262\zeta^2 + 188\zeta + 41)n + 2(11\zeta^4 + 53\zeta^3 + 79\zeta^2 + 47\zeta + 8).$$

This proves [MB, Conj. 6].

The functions $t^{(0,2n,0,0)}$ and $t^{(-1,2n+1,0,0)}$ seem to appear in connection with eigenvectors of the Hamiltonian of the XYZ chain [MB, RS, Z] and other spin chains [BH, H], though these relations have not yet been established rigorously. Partial results were obtained by Zinn-Justin [Z], who also derived recursions for these functions. One can give alternative proofs of those recursions using the relation to Painlevé tau functions. In fact, one can derive a general relation of the form

$$t^{(\mathbf{k}+2\mathbf{e}_1)} t^{(\mathbf{k}-2\mathbf{e}_1)} = A \left((2k_1+1)^2 \frac{d^2 t^{(\mathbf{k})}}{d\zeta^2} t^{(\mathbf{k})} - (2k_1+3)(2k_1-1) \left(\frac{dt^{(\mathbf{k})}}{d\zeta} \right)^2 \right) + B \frac{dt^{(\mathbf{k})}}{d\zeta} t^{(\mathbf{k})} + C (t^{(\mathbf{k})})^2.$$

The coefficients are more complicated than for (5.5), and we do not go into the details.

Using (4.5) and (4.7) in Proposition 2.5, we find that $t^{(k_0,k_1,k_2,k_3)}$ always satisfies a quadratic differential equation. This seems to be a new observation.

Proposition 5.5. *The polynomial $t = t^{(k_0,k_1,k_2,k_3)}(\zeta)$ satisfies a differential equation of the form*

$$\sum_{i \geq j \geq 0, i+j \leq 4} A_{ij} \frac{d^i t}{d\zeta^i} \frac{d^j t}{d\zeta^j} = 0 \quad (5.7)$$

with coefficients A_{ij} that are polynomials in ζ and k_0, \dots, k_3 .

One may normalize (5.7) so that

$$\begin{aligned} A_{40} &= e^3, & A_{31} &= -4e^3, & A_{22} &= 3e^3, \\ A_{30} &= 4\zeta^2(\zeta+1)^2(\zeta-1)^3(\zeta+2)^3(2\zeta+1)^4, \end{aligned}$$

where

$$e = \zeta(\zeta+1)(\zeta-1)(\zeta+2)(2\zeta+1).$$

The remaining coefficients depend on k_j and are too cumbersome to write down; for instance, A_{00} has 579 terms.

As an example, we consider the case of $t = t^{(0,2n,0,0)}$. It follows from (4.3) and (4.4g)–(4.4h) that

$$t^{(0,2n,0,0)}(\zeta) = \left(\frac{\zeta(\zeta+1)}{\zeta+2} \right)^{n(n-1)} f_n((2\zeta+1)^2),$$

with f_n a polynomial of degree $n(n-1)/2$. It is related to the polynomial q_n of [MB] by

$$q_n(z) = D_n z^{n(n-1)} f_n(z^{-2}),$$

where D_n is a constant, see [R3, §5.3]. In terms of $f_n(z)$, (5.7) takes the form

$$\begin{aligned} & z(z-1)^3(z-9)^3(f_n^{(4)}f_n - 4f_n^{(3)}f_n' + 3(f_n'')^2) \\ & + (7z-3)(z-1)^2(z-9)^3(f_n^{(3)}f_n - f_n''f_n') \\ & - 2(z-1)(z-9)\{(z+1)(z-9)^2n^2 + 2(z-9)^2n - 5z^3 + 105z^2 - 483z + 351\}f_n''f_n \\ & + 2(z-1)(z-9)\{(z+1)(z-9)^2n^2 + 2(z-9)^2n - z^3 + 9z^2 - 111z + 135\}(f_n')^2 \\ & - \{2(z-9)(z^3 - 39z^2 + 139z + 27)n^2 + 8(z-9)(3z^2 + 2z + 27)n \\ & - 2z^4 + 72z^3 - 876z^2 + 2184z - 1890\}f_n'f_n \\ & - 2n(n-1)\{(5z-21)(z-9)n^2 - (z+15)(z-9)n + z^2 + 22z + 9\}f_n^2 = 0. \end{aligned}$$

As a final remark, we stress that the functions $t_n^{(k_0, k_1, k_2, k_3)}$ are defined by explicit determinants. For instance, writing $a = 2\zeta + 1$, $b = \zeta/(\zeta + 2)$ and

$$G(x, y) = (\zeta + 2)xy(x + y) - \zeta(x^2 + y^2) - 2(\zeta^2 + 3\zeta + 1)xy + \zeta(2\zeta + 1)(x + y),$$

we have

$$\begin{aligned} t^{(n,n,0,0)} &= \lim_{\substack{x_1, \dots, x_n \rightarrow a \\ y_1, \dots, y_n \rightarrow b}} \frac{\prod_{i,j=1}^n G(x_i, y_j)}{\prod_{1 \leq i < j \leq n} (y_j - y_i)(x_j - x_i)} \det_{1 \leq i, j \leq n} \left(\frac{1}{G(x_i, y_j)} \right), \\ &= \frac{G(a, b)^{n^2}}{\prod_{j=1}^n (j-1)!^2} \det_{1 \leq i, j \leq n} \left(\frac{\partial^{i+j-2}}{\partial x^{i-1} \partial y^{j-1}} \Big|_{x=a, y=b} \frac{1}{G(x, y)} \right). \end{aligned} \quad (5.8)$$

These functions solve the recursion (5.6). This is reminiscent of how the Toda equation

$$\tau_{n+1}\tau_{n-1} = \tau_n''\tau_n - (\tau_n')^2 \quad (5.9)$$

is solved by Hankel determinants

$$\tau_n = \det_{1 \leq i, j \leq n} (f^{(i+j-2)}). \quad (5.10)$$

However, an important difference is that, whereas (5.9) is immediately obtained from (5.10) by applying the Jacobi–Desnanot identity, applying that identity to (5.8) leads to an equation involving x - and y -derivatives of $G(x, y)$, cf. [R2, Cor. 7.16]. The missing ingredient is the Schrödinger equation (or *quantum* Painlevé VI equation) of [R4], which allows us to express specialized x - and y -derivatives of G in terms of ζ -derivatives.

It should be mentioned that genuine Hankel determinants for tau functions of Painlevé VI have been given in [K]. These are quite different in nature from (5.8). It would be interesting to know whether identities such as (5.8) are peculiar to our choice of seed solution, or if similar formulas can be found for other solutions.

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DEPARTMENT OF MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG, SE-412 96 GÖTEBORG, SWEDEN

E-mail address: `hjalmar@chalmers.se`

URL: `http://www.math.chalmers.se/~hjalmar`